

Pairwise Coprime Nonrepresentable Integers for Two Coprime Bases

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Abstract

Let $p > q \geq 2$ be coprime integers. Call a positive integer (p, q) -nonrepresentable if it cannot be written as a sum of terms $p^i q^j$ for which no two distinct terms divide each other. Erdős and Lewin asked for which pairs there is an infinite sequence of pairwise coprime (p, q) -nonrepresentable integers. Yu and Chen answered this question except for the pairs $(5, 2)$, $(9, 2)$, and $(5, 3)$. We prove the remaining cases. The proof works by creating a criterion for generating such sequences, which can then be applied to the three pairs.

1 Introduction

Let $p > q \geq 2$ be coprime integers. Write

$$\mathcal{R}_{p,q} = \{p^i q^j : i, j \in \mathbb{N}\}.$$

A positive integer n is said to be (p, q) -representable if there is a finite subset $S \subseteq \mathcal{R}_{p,q}$ such that

$$n = \sum_{s \in S} s,$$

and no two distinct members of S divide one another. Otherwise, we call n (p, q) -nonrepresentable. $\mathcal{R}_{p,q}$ is said to be d -complete if every n is (p, q) -representable. Erdős and Lewin [2] initially proved that $\mathcal{R}_{p,q}$ is d -complete if and only if $\{p, q\} = \{2, 3\}$ and asked in the same study whether for every other pair, there are infinitely many pairwise coprime nonrepresentable n .

Yu and Chen [4, Corollary 1.2] proved that such a pairwise coprime family exists whenever

$$q > 3, \quad q = 3 \text{ and } p \neq 5, \quad q = 2 \text{ and } p \notin \{3, 5, 9\}.$$

In particular, excluding $\{p, q\} = \{2, 3\}$, we are left to classify

$$(p, q) = (5, 2), \quad (9, 2), \quad (5, 3).$$

We show that indeed there are infinitely many pairwise coprime nonrepresentable n for these cases.

Theorem 1. *For each*

$$(p, q) \in \{(5, 2), (9, 2), (5, 3)\},$$

there exists an infinite sequence of pairwise coprime (p, q) -nonrepresentable integers. Furthermore, every term of the sequence may be chosen so that it is coprime to pq .

Combining this with the result of Yu and Chen, we hence have the following:

Corollary 2. *Let $p > q \geq 2$ be coprime integers. There are infinitely many pairwise coprime (p, q) -nonrepresentable integers if and only if $\{p, q\} \neq \{2, 3\}$.*

2 Constructing Nonrepresentable Numbers

Consider the ordering induced on $\mathbb{N}^2$ by coordinatewise order

$$(i, j) \leq (i', j') \iff i \leq i' \text{ and } j \leq j'.$$

As p and q are coprime,

$$p^i q^j \mid p^{i'} q^{j'} \iff (i, j) \leq (i', j')$$

means that a valid representation corresponds exactly to a finite antichain $S \subseteq \mathbb{N}^2$ such that

$$n = \sum_{(i,j) \in S} p^i q^j.$$

We first show that representations are unique and simple when they have a clean modulo.

Lemma 3. *For $r, s \geq 1$, suppose $p^r < q^s$ and let*

$$S \subseteq \{0, \dots, r\} \times \{0, \dots, s-1\}$$

be an antichain. If

$$\sum_{(i,j) \in S} p^i q^j \equiv p^r \pmod{q^s},$$

then $S = \{(r, 0)\}$.

Proof. Define

$$N := \sum_{(i,j) \in S} p^i q^j.$$

As $N \equiv p^r \not\equiv 0 \pmod{q}$, S contains a point on the row $j = 0$. Denote this unique point by $(t, 0)$. We continue by a case split on t .

If $t = r$, the proof is immediate if S contains only a single element. Otherwise, choose a point $(i, j) \in S \setminus \{(r, 0)\}$ with minimal second coordinate j . However, $N - p^r \equiv p^i q^j \not\equiv 0 \pmod{q^{j+1}}$, which contradicts $q^s \mid N - p^r$. Hence $S = \{(r, 0)\}$ in this case.

For $t < r$, note first that having $(t, 0) \in S$ means no point has first coordinate r . For $0 \leq k \leq r$ define

$$N_k := \sum_{\substack{(i,j) \in S \\ i \geq k}} p^i q^j.$$

We will show by descending induction on k that

$$p^r - N_k = p^k X_k, \tag{3.1}$$

for some positive integer X_k . For $k = r$, this holds with $X_k = 1$. Suppose that (3.1) holds at $k + 1$. If S has no point with first coordinate k , then we can just set $X_k = pX_{k+1}$. Otherwise, write (k, j) to denote such a point.

Notice that as

$$p^r - N_k = p^{k+1} X_{k+1} - p^k q^j = p^k (pX_{k+1} - q^j),$$

setting $X_k = pX_{k+1} - q^j$ it suffices to show that $pX_{k+1} - q^j > 0$.

We first show $X_{k+1} \geq q^j$. Now, taking N modulo q^{j+1} ,

$$p^r \equiv N = N_{k+1} + p^k q^j + \sum_{\substack{(i',j') \in S \\ i' < k}} p^{i'} q^{j'} \equiv N_{k+1} + p^k q^j \pmod{q^{j+1}}.$$

Hence,

$$p^{k+1}X_{k+1} - p^k q^j \equiv 0 \pmod{q^{j+1}}.$$

Noting that p is invertible modulo q^{j+1} , we have

$$pX_{k+1} - q^j \equiv 0 \pmod{q^{j+1}}.$$

In particular, $q^j \mid pX_{k+1}$ and hence $q^j \mid X_{k+1}$. As X_{k+1} is positive, we have $X_{k+1} \geq q^j$. In the end, we have

$$pX_{k+1} - q^j \geq pq^j - q^j = (p-1)q^j > 0.$$

This completes the induction. Taking $k = 0$,

$$0 < p^r - N < q^s,$$

which contradicts $q^s \mid p^r - N$. So $t < r$ cannot occur. $\square$

Lemma 3 gives us a recipe to construct a (p, q) -nonrepresentable from a nonrepresentable number.

Lemma 4. *Let c be a positive integer that is not (p, q) -representable, and suppose that $\gcd(c, pq) = 1$. If $r, s \geq 1$ satisfy*

$$p^r < q^s, \tag{4.1}$$

$$cq^s < (p-1)p^r, \tag{4.2}$$

then $N = p^r + cq^s$ is (p, q) -nonrepresentable with $\gcd(N, pq) = 1$.

Proof. By (4.2), we have

$$N < p^r + (p-1)p^r = p^{r+1}.$$

Suppose for a contradiction that N has a representation corresponding to an antichain $S \subseteq \mathbb{N}^2$. Noting every $(i, j) \in S$ has $i \leq r$, alongside (4.1) we can apply Lemma 3 to see that the only point of S with $j < s$ is $(r, 0)$. Hence, we can write

$$c = \sum_{\substack{(i,j) \in S \\ j \geq s}} p^i q^{j-s}.$$

This clearly gives a representation of c , a contradiction. To see that $\gcd(N, pq) = 1$, if a prime $\mathfrak{p}$ divides q , then $N \equiv p^r \not\equiv 0 \pmod{\mathfrak{p}}$. If $\mathfrak{p}$ divides p , then $N \equiv cq^s \not\equiv 0 \pmod{\mathfrak{p}}$. $\square$

To satisfy the conditions of Lemma 4, we note the following elementary consequence that follows from the fact that fractional parts of irrational rotations are dense in $[0, 1]$.

Lemma 5. *Let $p, q > 1$ be coprime integers and let $\rho > 1$. For any positive integer n , there exist positive multiples r, s of n such that*

$$1 < \frac{q^s}{p^r} < \rho.$$

We can now give a sufficient condition to generate a sequence of pairwise nonrepresentables.

Theorem 6. *Let $p, q > 1$ be coprime integers. Suppose that there is a positive integer c with*

$$c < p - 1, \tag{6.1}$$

$$c \text{ is } (p, q)\text{-nonrepresentable}, \tag{6.2}$$

$$\gcd(c, pq) = 1, \tag{6.3}$$

$$\text{every prime divisor of } c + 1 \text{ divides } pq. \tag{6.4}$$

Then there is an infinite pairwise coprime sequence $(N_k)_{k \geq 0}$ such that every term of the sequence is (p, q) -nonrepresentable and coprime to pq .

Proof. We claim that for any $D \geq 1$ with $\gcd(D, pq) = 1$, there is a (p, q) -nonrepresentable number N such that $\gcd(N, Dpq) = 1$.

As p, q are units, we may take the smallest n such that $p^n \equiv q^n \equiv 1$ modulo D . Applying Lemma 5 with $\rho = \frac{p-1}{c} > 1$ and n , we get positive multiples r, s of n with

$$p^r < q^s < \frac{p-1}{c} p^r.$$

Taking $N = p^r + cq^s$, Lemma 4 shows us that N is (p, q) -nonrepresentable and coprime to pq . Furthermore, modulo D , $N \equiv 1 + c$. Now, as D is coprime to pq , along with (6.4) we get $\gcd(c+1, D) = 1$, and hence $\gcd(N, D) = 1$. This proves our claim.

Set $D_0 = 1$, and let N_0 be the number obtained via the claim. After $N_0, \dots, N_{k-1}$ have been chosen, apply the claim with $D_k = N_0 N_1 \cdots N_{k-1}$. N_k is coprime to D_k , hence to all preceding terms. We can therefore inductively produce our sequence. $\square$

We can use Theorem 6 directly to prove Theorem 1. We will first give a short lemma about nonrepresentability.

Lemma 7. *If $0 < c < p$ and c is not a power of q , then c is (p, q) -nonrepresentable.*

Proof. If c has a representation, $c < p$ forces $i = 0$. As representations are antichains, this forces $c = q^j$, a contradiction. $\square$

Proof of Theorem 1. For $(p, q) = (5, 2)$, take $c = 3$; for $(9, 2)$, take $c = 5$; and for $(5, 3)$, take $c = 2$. In all cases, using Lemma 7, it can easily be checked that we satisfy conditions (6.1)–(6.4); hence, we can apply Theorem 6. $\square$

3 Formal verification and AI Usage

The results above have been formalised in Lean 4 using the mathlib library [1, 3]. The exact statement proven is slightly stronger, where we give a function $f: \mathbb{N} \rightarrow \mathbb{N}$ whose values are pairwise coprime and are greater than 1, (p, q) -nonrepresentable, and coprime to pq . The source code is available as a public repository at <https://github.com/Apiros3/erdos1110/tree/arxiv-v1>.

The proof for this work has been produced with the assistance of GPT-5.6 and GPT-5.5, and the Lean formalisation was fully generated by GPT-5.6. This manuscript has been produced completely by human (modulo spelling / syntax).

References

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